

Quantifying entanglement of maximal dimension in bipartite mixed states

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The Schmidt coefficients capture all entanglement properties of a pure bipartite state and therefore determine its usefulness for quantum information processing. While the quantification of the corresponding properties in mixed states is important both from a theoretical and a practical point of view, it is considerably more difficult, and methods beyond estimates for the concurrence are elusive. In particular this holds for a quantitative assessment of the most valuable resource, the forms of entanglement that can only exist in high-dimensional systems. We derive a framework for lower bounding the appropriate measure of entanglement, the so-called *G-concurrence*, through few local measurements. Moreover, we show that these bounds have relevant applications also for multipartite states.

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Understanding the nature and operational uses of entanglement constitutes one of the key challenges of quantum information theory. While most algorithms that allow for a provable advantage with respect to classical computation exhibit this ubiquitous feature of quantum systems, it is not entirely clear whether it actually is required for the promising field of quantum computation and simulation to outperform their respective classical counterparts.

Consequently much effort has been invested in understanding the interplay between entanglement structure and resource properties of multipartite quantum states [1–4]. One of the key results for computing with pure quantum states is the fact that, in order to go beyond the classical realm, a large dimension of entanglement is required while indeed any actual continuous measure of entanglement can be rather small [5]. Whether or how this statement translates to realistic conditions, *i.e.*, mixed-state quantum computing, is not at all clear. Here one could imagine a speed-up without any entanglement present at all, or, on the contrary, the need for high-dimensional entanglement in a more robust sense. However, it appears intuitively clear that mixed states with substantial overlap to states, whose resource content is exponentially hard to simulate classically, continue to be sufficient.

To answer such questions and to ultimately gain a deeper insight into the very nature of entanglement one would need a thorough quantification of all possible features of mixed-state entanglement. The sheer complexity of this task makes general solutions unlikely (recall that

even deciding whether or not a given state is entangled is an NP-hard problem [6]).

A first interesting step in this direction could be the quantitative characterization of high-dimensional entanglement, *i.e.*, the most expensive resource in bipartite systems. One of the paradigmatic measures for the dimensionality of entanglement is the Schmidt number of mixed quantum states [7], for which various methods of certification exist [8–10]. However, the Schmidt number in itself is not entirely significant, as even the highest possible dimensionality can lie in the vicinity of completely separable states [11]. A robust quantification of mixed-state entanglement dimensionality can be made by using continuous measures of entanglement dimension which possibly bear also an operational meaning, beyond the question of mere computability. For bipartite entangled states the natural candidate for this purpose is the family of concurrence monotones introduced by Gour [12]. For a $d \times d$ -dimensional system, there are $d - 1$ such monotones $k = 2, \dots, d$. The k th concurrence monotone (which we will call for short k -concurrence) vanishes for a given state if its Schmidt number does not exceed $k - 1$. The usual concurrence [13, 14] coincides with the 2-concurrence in this family (up to a normalization constant). The measure for $k = d$ quantifies to which extent the maximum Schmidt number is contained in a state and is usually termed *G-concurrence*. While there exist various bounds for the usual concurrence [15–21], there are no mixed-state bounds for any of the other concurrences monotones, in particular not for the *G-concurrence*. Such a bound would go beyond

giving an answer to the question whether or not a state contains entanglement of maximum dimensionality.

This is exactly what we achieve in this article: We first derive a general method how this measure can be efficiently lower bounded by using nonlinear witness techniques, allowing for a mixed-state quantification of G -concurrence in an experimentally feasible way. Furthermore, we find the exact solution for the G -concurrence of the so-called axisymmetric states [22], a highly symmetric two-parameter family of $d \times d$ mixed states. This solution provides the basis for a simple method to find lower bounds to the G -concurrence of arbitrary mixed states. As a byproduct, it also allows us to find lower bounds to the distance between the state of interest and the set of separable states, and, more generally, to any set of states with bounded Schmidt number.

Nonlinear G -concurrence witness.— We commence by a brief definition of the relevant concepts, before going to our first main theorem. For pure quantum states $|\psi\rangle \in \mathbb{C}_A^d \otimes \mathbb{C}_B^d$, $|\psi\rangle = \sum_{jk} c_{jk} |jk\rangle$ the G -concurrence is defined as the d th root of the product of the d eigenvalues of the marginal [12]. Denoting the Schmidt coefficients of the state as $\lambda_j \geq 0$ (i.e., $|\psi\rangle = \sum_j \lambda_j |a_j b_j\rangle$; consequently the marginal eigenvalues are λ_j^2), we can define

$$C_G(|\psi\rangle) := d(\lambda_1 \lambda_2 \cdots \lambda_d)^{\frac{2}{d}}, \quad (1)$$

so that $0 \leq C_G(|\psi\rangle) \leq 1 \forall |\psi\rangle$. The extension to mixed states is straightforward via the convex roof [23]

$$C_G(\rho) := \min_{\{p_k, \psi_k\}} \sum_j p_j C_G(|\psi_j\rangle), \quad (2)$$

where the minimum is taken over all pure-state decompositions $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$.

The idea here is to derive a tight lower bound for the G -concurrence of pure states in a form that admits a straightforward extension to a nonlinear witness lower bound for mixed states. In spirit this work follows Refs. [17, 24, 25], that is, if a state does *not* belong to a certain entanglement class, the modulus of the off-diagonal elements cannot exceed a certain monotonically increasing function of the diagonal elements. Using elementary algebra we arrive at our first main result (the proof is given in the Appendix),

$$C_G(\rho) \geq B_G(\rho) = \sum_{j,k} ((d-2)\delta_{jk} + 1) |\langle jj|\rho|kk\rangle| - d \sum_{\sigma \neq \mathbf{1}} \left(\text{Tr} \left(\bigotimes_{j=0}^d |j\sigma(j)\rangle\langle j\sigma(j)| \rho^{\otimes d} \right)^{\frac{1}{d}} \right), \quad (3)$$

where $\sum_{\sigma \neq \mathbf{1}}$ denotes the sum over all permutations of the levels of party B , excluding the identical permutation. This general lower bound is both surprisingly simple and transparent. It is expressed via density matrix elements and requires the knowledge of only

$O(d^2)$ out of the $d^4 - 1$ elements. While Eq. (3) is written in terms of $d \times d$ -dimensional systems, it is obvious that the bound can directly be applied also to the different bipartitions of multipartite systems, as we will see in the example below. Before we proceed with a more detailed discussion let us briefly comment that the bound (3) is tight at least for all maximally entangled states of dimension d , i.e., $B_G(|\Phi_d\rangle) = 1$, where $|\Phi_d\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle$.

By investigating the noise resistance we find that the worst possible kind of noise for our bound is white noise, as it maximally affects the negative terms in the bound. For dimension $d = 3$, e.g., we can study the white-noise tolerance by considering the state $\rho(p) = p|\Phi_d\rangle\langle\Phi_d| + \frac{1-p}{d^2} \mathbb{1}_{d^2}$. Inserting this state into our bound we find that it can reveal the presence of 3-concurrence down to $p = \frac{2}{3}$ (which is close to the exact value $p = \frac{5}{8}$, see below). However, for higher dimensions the noise resistance of the G -concurrence decreases rapidly. Possibly the quality of the bound (3) can be improved by finding different estimates of the G -concurrence for pure states.

Interestingly, the nonlinear witness Eq. (3) is not the only possibility to estimate the quality of high-dimensional entanglement in a mixed bipartite state. In the following, we first describe the exact solution of $C_G(\rho)$ for certain symmetric states. By means of this solution we can achieve an independent lower bound for arbitrary $d \times d$ states.

Exact solution for axisymmetric states.— Families of highly symmetric states often allow for an exact solution of entanglement-related problems [27, 28, 46]. Here we consider the axisymmetric states, a two-parameter family of $d \times d$ -dimensional mixed states [20, 22]. They comprise all mixed states that have the same symmetries as $|\Phi_d\rangle$, that is, (i) permutation symmetry of the qudits, (ii) invariance under simultaneous exchange of two levels for both parties, that is, $|j\rangle_A \leftrightarrow |k\rangle_A$, $|j\rangle_B \leftrightarrow |k\rangle_B$, and (iii) symmetry under simultaneous local phase rotations

$$V(\varphi_1, \varphi_2, \dots, \varphi_{n-1}) = e^{i \sum \varphi_j \mathbf{g}_j} \otimes e^{-i \sum \varphi_j \mathbf{g}_j}.$$

Here, \mathbf{g}_j are the $(d-1)$ diagonal generators of $\text{SU}(d)$.

The axisymmetric states can be written as mixtures of three states

$$\begin{aligned} \rho^{\text{axi}} &= p |\Phi_d\rangle\langle\Phi_d| + (1-p) [q \tilde{\rho}_1 + (1-q) \tilde{\rho}_2] \quad , \\ \tilde{\rho}_1 &= \frac{1}{d^2} \mathbb{1}_{d^2} - |\Phi_d\rangle\langle\Phi_d| \quad , \\ \tilde{\rho}_2 &= \frac{1}{d(d-1)} \sum_{j \neq k} |jk\rangle\langle jk| \end{aligned} \quad (4)$$

where $0 \leq p, q \leq 1$. They can be represented by a triangle. Remarkably it was found that Schmidt-number related entanglement properties are affine functions of the fidelity of ρ^{axi} with the maximally entangled state

$F = \text{Tr}(\rho^{\text{axi}} |\Phi_d\rangle\langle\Phi_d|)$. For example, the borders of the Schmidt-number classes are lines of constant fidelity F (for $F \geq \frac{1}{d}$). In Ref. [20] it was shown that also the 2-concurrence of ρ^{axi} is an affine function of F , namely $C_2(\rho^{\text{axi}}) = \sqrt{\frac{2d}{d-1}} (F - \frac{1}{d})$ for $F \geq \frac{1}{d}$. By using the methods from Refs. [46] and [27] we show that the exact G -concurrence for axisymmetric states is

$$C_G(\rho^{\text{axi}}) = \max[1 - d(1 - F), 0] . \quad (5)$$

In order to prove Eq. (5), one first notes that for symmetric mixed states it suffices to minimize C_G for pure states ψ as a function of the Schmidt coefficients under the constraint of fixed fidelity $F = |\langle\psi|\Phi_d\rangle|^2 = \text{Tr}(\rho^{\text{axi}} |\psi\rangle\langle\psi|)$ and to convexify the resulting function (cf. Ref. [27])

$$C_G(\rho^{\text{axi}}) = \text{co}C_G(\psi) . \quad (6)$$

(here, $\text{co}C_G(\psi)$ denotes the convex hull). In complete analogy with the approach in Ref. [46] one finds that the problem effectively depends only on a *single* parameter, the fidelity F :

$$C_G(F) = d(\alpha\beta^{d-1})^{\frac{2}{d}} , \quad (7)$$

where

$$\alpha = \frac{1}{\sqrt{d}} \left(\sqrt{F} - \sqrt{d-1} \sqrt{1-F} \right) , \quad F \geq \frac{d-1}{d} ,$$

$$\beta = \frac{1}{\sqrt{d}} \left(\sqrt{F} + \frac{\sqrt{1-F}}{\sqrt{d-1}} \right) .$$

In the Appendix we present more details of this derivation. Moreover, we prove that the function in Eq. (7) is concave such that its convex hull is the affine function (5). We show the result in Fig. 1 for $d = 4$.

Arbitrary states.— The exact solution for axisymmetric states is interesting not only from a mathematical point of view. We can use it to obtain a lower bound on $C_G(\rho)$ for arbitrary states ρ by noting that the average over all the operations in the group of axisymmetries \mathcal{V} (“twirling”) applied to ρ represents a projection

$$\mathbb{P}^{\text{axi}}(\rho) = \int d\mathcal{V} \mathcal{V} \rho \mathcal{V}^\dagger$$

into the axisymmetric states [27]. On the other hand, averaging over the operations \mathcal{V} can only *reduce* the entanglement, so that for $\rho^{\text{axi}}(\rho) := \mathbb{P}^{\text{axi}}(\rho)$ we have a lower bound [1, 29]

$$C_G[\rho^{\text{axi}}(\rho)] \leq C_G(\rho) . \quad (8)$$

This bound, which explicit form is given in Eq. (5), has been recently proven to hold also for fidelity parameters

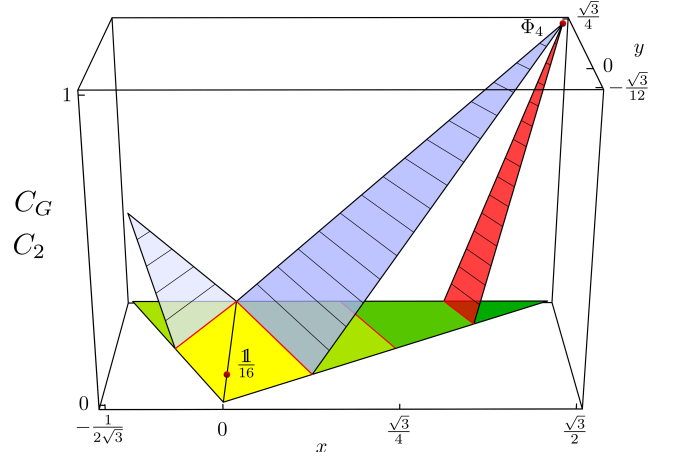


FIG. 1. The G -concurrence (red) and the 2-concurrence (light blue) for 4×4 axisymmetric states. The 2-concurrence (for pure states) is defined here as $C_2 = \sqrt{\frac{d}{d-1} (1 - \text{Tr} \rho_A^2)}$, where ρ_A denotes the reduced state of party A . In the plane we show the axisymmetric states and the borders between the Schmidt-number classes (red solid lines) which, for $x > 0$ are lines of constant fidelity $F = \text{Tr}(\rho^{\text{axi}} |\Phi_d\rangle\langle\Phi_d|)$. Here, x and y are the appropriate coordinates to parametrize the axisymmetric states in a geometry that corresponds to the Hilbert-Schmidt metric [20, 22].

taken with respect to arbitrary states [30]. The components of the symmetrized state $\rho^{\text{axi}}(\rho)$ are easily obtained via the relations

$$\rho_{jk, jk}^{\text{axi}} = \frac{1}{d} \left[\delta_{jk} \sum_a \rho_{aa, aa} + \frac{1 - \delta_{jk}}{d-1} \sum_{a \neq b} \rho_{ab, ab} \right] \quad (9a)$$

$$\rho_{jk, lm}^{\text{axi}} = \frac{\delta_{jk} \delta_{lm} (1 - \delta_{jl})}{d(d-1)} \sum_{a > b} (\rho_{aa, bb} + \rho_{bb, aa}) . \quad (9b)$$

The symmetrization requires some care since one may lose all the entanglement by inappropriately choosing the local bases. By exploiting local unitary invariance of the G -concurrence, $C_G([U_A \otimes U_B] \rho [U_A \otimes U_B]^\dagger) = C_G(\rho)$ we may improve the bound by finding the best local unitaries before doing the projection (8) so as to achieve the largest $C_G[\rho^{\text{axi}}(\rho)]$. Clearly, this holds as well for the bound in Eq. (3).

Indeed, both the bounds (3) and (8) can be improved even further by observing that the G -concurrence is an $\text{SL}(d, \mathbb{C})^{\otimes 2}$ invariant [3]. According to Verstraete *et al.* [31] an entanglement monotone based on a local SL invariant is maximized on the so-called normal form of the state. The hallmark of the normal form is that it has maximally mixed local density matrices [31, 32]. It can be found via an algorithm described in [31]. Thus, an exact solution (or lower bound) for a local SL invariant like C_G over a family of symmetric states can be used to calculate a lower bound of $C_G(\rho)$ for arbitrary states ρ by the following procedure [33]:

- find the normal form $\rho^{\text{NF}}(\rho)$ (in general not normalized to 1; re-normalization is not necessary because of the homogeneity of C_G of degree 1 in the density matrix); if the normal form vanishes the procedure terminates and $C_G(\rho) = 0$;
- apply optimal local unitaries to $\rho^{\text{NF}}(\rho)$ (as described above) which leads to $\tilde{\rho}^{\text{NF}}$ and do the projection $\mathbb{P}^{\text{axi}}(\tilde{\rho}^{\text{NF}}) =: \tilde{\rho}_{\text{axi}}^{\text{NF}}$;
- read off the bound for the G -concurrence from this state

$$C_G(\rho) \geq C_G(\tilde{\rho}_{\text{axi}}^{\text{NF}}) . \quad (10)$$

Clearly, in order to produce the normal form knowledge of all the matrix elements of ρ is required. Hence improving the bounds via $\text{SL}(d, \mathbb{C})^{\otimes 2}$ (as well as via unitary) optimization is more expensive with respect to the experimental effort.

To conclude this section, we show how our results can also be used in some cases to guarantee that an arbitrary state ρ has a finite distance to any set of states with bounded Schmidt number—in particular to the set of separable states—, thus rendering the computed bounds of $C_G(\rho)$ more meaningful and robust measures of the entanglement of ρ . Let \mathcal{S}_k be the set of all states with Schmidt number $k < d$, $\mathcal{S}_k^{\text{axi}}$ the set of Schmidt number k states in the axisymmetric family, and ρ^{axi} the symmetrization of ρ . Then, the following inequality holds [34]:

$$\min_{\sigma \in \mathcal{S}_k^{\text{axi}}} \|\rho^{\text{axi}} - \sigma\|_{\text{HS}} \leq \min_{\sigma \in \mathcal{S}_k} \|\rho - \sigma\|_{\text{HS}} . \quad (11)$$

This inequality tells us that, given ρ , whenever its projection ρ^{axi} lies at a finite distance with respect to the closest axisymmetric state with Schmidt number k , we know that the distance between ρ and the closest Schmidt number k state is at least as large. A consequence of this is that any nonzero value of the bound $C_G[\rho^{\text{axi}}(\rho)]$ rules out the possibility of ρ being arbitrarily close to a separable state.

Application of the bounds to multipartite states.— While the usefulness of our bounds for the characterization of bipartite states is apparent, we would like to point out that this is true also in the context of multipartite states. To this end, let us consider a four-qubit cluster state

$$|\text{Cl}_{ABCD}\rangle = \frac{1}{2}(|0000\rangle + |0111\rangle + |1011\rangle + |1100\rangle) .$$

Each of the bipartitions $(AB)(CD)$, $(AC)(BD)$, and $(AD)(BC)$ may be regarded as a 4×4 system where the state of $(AB)(CD)$ is of Schmidt rank 2 and the others have Schmidt rank 4. Indeed, for the latter bipartitions the state is locally equivalent to $|\Phi_4\rangle$ and has maximal G -concurrence, *e.g.*, $C_G(|\text{Cl}_{(AC)(BD)}\rangle) = 1$.

Now we may ask how this resource behaves when noise is added to $|\text{Cl}_{ABCD}\rangle$. We use the white-noise tolerance of the G -concurrence on a rank-4 bipartition as a model to answer this question. Physically, this means we ask up to which admixture w_4 of white noise any decomposition of the resulting state $\rho_{ABCD} = (1 - w)|\text{Cl}_{ABCD}\rangle\langle\text{Cl}_{ABCD}| + \frac{w}{16}\mathbb{1}_{16}$ contains a state of Schmidt rank 4, *e.g.*, on the bipartition $(AC)(BD)$. The corresponding fidelity is $F_4 = \frac{3}{4}$ so that $w_4 = \frac{4}{15}$. This is a remarkable result, as it has to be contrasted with the noise tolerance of genuine multipartite entanglement (GME) for this state, $w_4^{\text{GME}} = \frac{8}{13}$ (cf. Ref. [35]). It shows, as expected, that a well-specified resource of multipartite entanglement behaves differently from GME.

This discussion is straightforwardly extended to linear cluster states of large (even) number N of qubits. The existence of bipartitions with full Schmidt rank is one of their important properties [36]. In that case, the maximum Schmidt rank across the bipartitions is $d = 2^{(N/2)}$, and hence $w_N \simeq 2^{-(N/2)}$. Recall that for linear N -qubit cluster states $w_N^{\text{GME}} \simeq 1 - (N/3)2^{-(N/3)}$ [37]. That is, while the noise tolerance of GME in large linear cluster states is practically perfect, the maximum Schmidt-rank resource becomes exponentially fragile with increasing N .

Conclusions.— We have presented two independent quantification methods for high-dimensional entanglement, *i.e.*, of the resource characterized by the maximum number of non-vanishing Schmidt coefficients, in bipartite mixed states. This is achieved by estimates of the G -concurrence via a nonlinear witness on the one hand, and by an exact solution for axisymmetric states on the other hand. Our nonlinear witness Eq. (3) extends the possibility to detect entanglement of Schmidt number 2 [25] to maximum Schmidt number d . At the same time, this nonlinear witness (3), as well as the projection witness (8), is quantitative [38] and can be experimentally determined by measuring a number of observables of order d^2 which is considerably smaller than $d^4 - 1$, the number of all parameters of the state. This shows that the developed methods are suitable also to provide a quantitative analysis of recent efforts at producing high-dimensionally entangled states in the lab. The fact that Schmidt numbers equal to the system dimension were certified *e.g.* in Refs. [39–42] implies that the respective G -concurrence will be nonzero and the data taken should suffice to apply our methods. Due to the possibility of $\text{SL}(d, \mathbb{C})^{\otimes 2}$ optimization, entanglement detection through our approach is superior compared to merely using an optimal Schmidt number witness. However, exploiting this possibility requires complete knowledge of the state parameters. Moreover, we have outlined how our methods can be applied also in the investigation of multipartite entanglement. We have shown that the resource of maximum Schmidt number across the bipartitions of N -qubit cluster states is exponentially fragile with respect to the admixture of white noise. In addi-

tion, we mention that, in principle, it is possible to define a genuine multipartite G -concurrence in analogy with Ref. [17] in order to quantitatively describe the Schmidt-number vectors of a multipartite system [43, 44]. Finally, we note that similar techniques to the ones we develop in the first part could potentially be used to lower bound any quantity that can be expressed as a polynomial of state coefficients, such as other SL invariants [3, 45].

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APPENDIX

The nonlinear witness for the G -concurrence

Given an arbitrary pure state $|\psi\rangle = \sum_{i,j} c_{ij} |ij\rangle$, its G -concurrence can be computed as

$$C_G(|\psi\rangle) = d(\lambda_1 \lambda_2 \cdots \lambda_d)^{\frac{2}{d}} = d|\det c|^{\frac{2}{d}}, \quad (12)$$

where the λ_i are its Schmidt coefficients (i.e. $|\psi\rangle = \sum_i \lambda_i |\alpha_i \beta_i\rangle$), and c is a $d \times d$ matrix with elements c_{ij} . We can use the triangle inequality twice to lower bound the determinant as

$$|\det c| = \left| \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^d c_{i\sigma(i)} \right| \quad (13)$$

$$\geq \left| \prod_{i=1}^d c_{ii} \right| - \left| \sum_{\sigma \neq \mathbb{1}} \text{sgn}(\sigma) \prod_{i=1}^d c_{i\sigma(i)} \right| \quad (14)$$

$$\geq \left| \prod_{i=1}^d c_{ii} \right| - \sum_{\sigma \neq \mathbb{1}} \left| \prod_{i=1}^d c_{i\sigma(i)} \right|, \quad (15)$$

where $\sigma = \mathbb{1}$ is the identity permutation. Now, let us rename $|\prod_{i=1}^d c_{ii}| \equiv X$ and $\sum_{\sigma \neq \mathbb{1}} |\prod_{i=1}^d c_{i\sigma(i)}| \equiv Y$. For any two positive numbers X, Y , it is immediate to check that

$$f \equiv X^{\frac{2}{d}} - Y^{\frac{2}{d}} \leq |X - Y|^{\frac{2}{d}} \equiv g.$$

Indeed, if $X < Y$, the inequality is trivially satisfied. On the other hand, if $X \geq Y \geq 0$, we just have to look at

the convexity of f and g . At the extreme points of this interval, i.e. when $Y = 0, X$, the inequality is saturated. To see what happens in between, we compute the second derivatives of f and g with respect to Y , for an arbitrary X :

$$f'' \equiv \frac{d^2 f}{dY^2} = \frac{2}{d} \left(1 - \frac{2}{d}\right) Y^{\frac{2}{d}-2}, \quad (16)$$

$$g'' \equiv \frac{d^2 g}{dY^2} = -\frac{2}{d} \left(1 - \frac{2}{d}\right) (X - Y)^{\frac{2}{d}-2}. \quad (17)$$

We readily see that $f'' \geq 0$ and $g'' \leq 0$ for $d \geq 2$, which means that f is convex and g is concave, thus $f \leq g$ and we can write

$$|\det c|^{\frac{2}{d}} \geq \left| \prod_{i=1}^d c_{ii} \right|^{\frac{2}{d}} - \sum_{\sigma \neq \mathbb{1}} \left| \prod_{i=1}^d c_{i\sigma(i)} \right|^{\frac{2}{d}}. \quad (18)$$

It will prove useful to further lower the bound by replacing the positive term in the r.h.s. of Eq. (18) by a bilinear function of the coefficients c_{ii} , namely

$$\left| \prod_{i=1}^d c_{ii} \right|^{\frac{2}{d}} \geq \alpha \sum_{i \neq j} c_{ii} c_{jj}^* - \beta \sum_{i=1}^d |c_{ii}|^2 \quad (19)$$

for some real coefficients α and β . In order to prove this new inequality, we begin by rewriting it as

$$\left| \prod_{i=1}^d c_{ii} \right|^{\frac{2}{d}} \geq \alpha \left| \sum_{i=1}^d c_{ii} \right|^2 - (\alpha + \beta) \sum_{i=1}^d |c_{ii}|^2. \quad (20)$$

Note that the modulus makes it completely independent on complex phases, hence we can consider the coefficients c_{ii} to be real for the rest of this proof. Furthermore, the inequality is scale invariant, so we deliberately fix $\sum_{i=1}^d c_{ii}^2 = 1$ and, once again, we rewrite

$$\begin{aligned} P(\{c_{ii}\})^{\frac{2}{d}} - \alpha S^2 + \alpha + \beta \\ \geq P_{\min}(S)^{\frac{2}{d}} - \alpha S^2 + \alpha + \beta \geq 0, \end{aligned} \quad (21)$$

where we have defined $P(\{x_i\}) \equiv \prod_{i=1}^d x_i$, and $P_{\min}(S) = \min_{\{x_i\}: \sum_i x_i^2=1; \sum_i x_i=S} P(\{x_i\})$ for a set of d arbitrary parameters $\{x_i\} \in [0, 1]^d$. It is clear that $P_{\min}(S \leq \sqrt{d-1}) = 0$, since one can choose $x_d = 0$ and still find a set $\{x_i\}_{i=1}^{d-1}$ that fulfils the required conditions. As we increase the value of S above this threshold, the minimum should still be attained when x_d is as close to zero as possible. Such minimal value of x_d is directly obtained by solving the simpler minimization $\min_{\{x_i\}} x_d$, subject to the original constraints. This is a straightforward calculation via Lagrange multipliers. The corresponding Lagrangian is

$$\mathcal{L} = x_d - \lambda \left(\sum_{i=1}^d x_i - S \right) + \frac{\mu}{2} \left(\sum_{i=1}^d x_i^2 - 1 \right),$$

where λ, μ are Lagrange multipliers, and its symmetry already tells us that all the $x_{i < d}$ have to be equally valued. We find $x_i = \lambda/\mu$ for $i < d$, and $x_d = (\lambda - 1)/\mu$. The multipliers are

$$\lambda = \frac{\mu S + 1}{d}, \quad \mu = \pm \sqrt{\frac{d-1}{d-S^2}}.$$

We see from the form of μ that $S \leq \sqrt{d}$. In terms of S and d , an extreme value of $\prod_i x_i$ is attained when the coefficients are

$$x_{i < d} = \frac{S \pm \sqrt{(d-S^2)/(d-1)}}{d}, \quad (22)$$

$$x_d = \frac{S \mp \sqrt{(d-S^2)(d-1)}}{d}. \quad (23)$$

The first solution corresponds to the minimum.

We now prove Eq. (21) by solving for α and β the more restrictive set of inequalities

$$-\alpha S^2 + \alpha + \beta \geq 0, \quad \forall S < \sqrt{d-1} \quad (24)$$

$$P_L(S) - \alpha S^2 + \alpha + \beta \geq 0, \forall \sqrt{d-1} \leq S \leq \sqrt{d}, \quad (25)$$

where $P_L(S)$ is a linear lower bound of $P_{\min}(S)^{\frac{2}{d}}$ that is tight at the extreme values of the interval $\sqrt{d-1} \leq S \leq \sqrt{d}$. The function $P_L(S)$ exists if $P_{\min}(S)^{\frac{2}{d}}$ is a fully concave function. One can readily check that this is indeed the case, for the equation

$$P''_{\min}(S) \equiv \frac{d^2(P_{\min}(S)^{\frac{2}{d}})}{dS^2} = 0$$

has no solution in the relevant domain, hence there are no inflection points. Then one only has to observe the sign of an intermediate point, e.g. $P''_{\min}(\sqrt{d-1/2})$. Such function approaches zero exclusively in the asymptotic limit $d \rightarrow \infty$, and it is negative for any other (finite) value of $d \geq 3$, hence $P''_{\min}(\sqrt{d-1/2})$ is always negative and therefore $P_{\min}(S)^{\frac{2}{d}}$ is a concave function of S , for any d . Taking into account that $P_{\min}(S \leq \sqrt{d-1}) = 0$ and $P_{\min}(\sqrt{d}) = 1/d$, we may then write $P_L(S)$ as

$$P_L(S) = \frac{S - \sqrt{d-1}}{d(\sqrt{d} - \sqrt{d-1})}. \quad (26)$$

Finding values of α and β such that Eqs. (24) and (25) hold is straightforward. By, e.g., demanding that Eq.(25) be tight for $S = \sqrt{d}$, we get rid of one parameter. We obtain $\beta = \alpha(d-1) - 1/d$. Then, imposing that Eq. (24) be tight for $S = \sqrt{d-1}$ yields $\alpha = 1/d$, and thus $\beta = 1 - 2/d$. With these values of α and β , we can guarantee that Eq. (21) is satisfied for all S .

Summing up, we use Eqs. (18) and (19) to lower-bound the G -concurrence of an arbitrary bipartite pure state $|\psi\rangle$ as $C_G(|\psi\rangle) \geq C_G^{\downarrow}(|\psi\rangle)$, where

$$C_G^{\downarrow}(|\psi\rangle) = \sum_{i \neq j} c_{ii} c_{jj}^* - (d-2) \sum_{i=1}^d |c_{ii}|^2 - d \sum_{\sigma \neq \mathbf{1}} \left| \prod_{i=1}^d c_{i\sigma(i)} \right|^{\frac{2}{d}}. \quad (27)$$

The lower bound for the convex roof extension to mixed states, $C_G(\rho)$, hence follows:

$$\begin{aligned} C_G(\rho) &= \min_{\{p_k, \psi_k\}} \sum_k p_k C_G(|\psi_k\rangle) \geq \min_{\{p_k, \psi_k\}} \sum_k p_k C_G^{\downarrow}(|\psi_k\rangle) \\ &\geq \min_{\{p_k, \psi_k\}} \sum_k p_k \sum_{i \neq j} c_{ii}^k c_{jj}^{k*} - \max_{\{p_k, \psi_k\}} \sum_k p_k \left((d-2) \sum_{i=1}^d |c_{ii}^k|^2 + d \sum_{\sigma \neq \mathbf{1}} \left| \prod_{i=1}^d c_{i\sigma(i)}^k \right|^{\frac{2}{d}} \right) \\ &= \sum_{i \neq j} \langle ii | \rho | jj \rangle - (d-2) \sum_{i=1}^d \langle ii | \rho | jj \rangle - d \sum_{\sigma \neq \mathbf{1}} \max_{\{p_k, \psi_k\}} \sum_k p_k \left| \prod_{i=1}^d c_{i\sigma(i)}^k \right|^{\frac{2}{d}} \\ &\geq \sum_{i \neq j} \langle ii | \rho | jj \rangle - (d-2) \sum_{i=1}^d \langle ii | \rho | jj \rangle - d \sum_{\sigma \neq \mathbf{1}} \left(\prod_{i=1}^d \langle i\sigma(i) | \rho | i\sigma(i) \rangle \right)^{\frac{1}{d}}, \end{aligned} \quad (28)$$

where for the last inequality we have used the subadditivity of the root function to write

$$\begin{aligned} \sum_k p_k \left| \prod_{i=1}^d c_{i\sigma(i)}^k \right|^{\frac{2}{d}} &\leq \left(\sum_k p_k \prod_{i=1}^d |c_{i\sigma(i)}^k|^2 \right)^{\frac{1}{d}} \\ &\leq \left(\prod_{i=1}^d \sum_k p_k |c_{i\sigma(i)}^k|^2 \right)^{\frac{1}{d}}. \end{aligned} \quad (29)$$

G-concurrence of axisymmetric states

Derivation of Eq. (7)

As mentioned in the main text, the proof proceeds through a minimization of the entanglement measure under consideration (here the G -concurrence) on pure states first. Consider therefore $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ with its Schmidt decomposition $|\psi\rangle = \sum_{j=0}^d \lambda_j |a_j b_j\rangle$. The minimization of $C_G(|\psi\rangle)$ is under the condition that the fidelity of $|\psi\rangle$ with the maximally entangled state $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^d |jj\rangle$ be fixed, $|\langle \Phi_d | \psi \rangle|^2 = F_\psi$.

We use a fact noted by Terhal and Vollbrecht [46] that the largest value of the fidelity $F_\psi = |\langle \Phi_d | \psi \rangle|^2$ is obtained if the Schmidt bases $\{a_j\}$, $\{b_k\}$ coincide with the computational basis. With this choice of bases, the only remaining parameters are the Schmidt coefficients λ_j , and

$$\sum_j \lambda_j = \sqrt{d F_\psi} . \quad (30)$$

We are interested in non-vanishing $C_G(|\psi\rangle)$, therefore we can assume $\lambda_j \neq 0$. Further, we have the normalization

$$\sum_j \lambda_j^2 = 1 . \quad (31)$$

Since $x^{1/d}$ is monotonous, we can just minimize the product $\lambda_1^2 \lambda_2^2 \cdots \lambda_d^2$. By introducing Lagrange multipliers A and B for the conditions above, we arrive at the equations

$$\lambda_j \left(\prod_{n \neq j} \lambda_n^2 - B \right) = A , \quad j = 1, \dots, d . \quad (32)$$

We can use any two of the Eqs. (32) to obtain

$$(\lambda_j - \lambda_k) \left(\lambda_j \lambda_k \prod_{n \neq j, k} \lambda_n^2 - B \right) = 0 , \quad (33)$$

which can be satisfied if $\lambda_j = \lambda_k$ or $\lambda_j \lambda_k \prod_{n \neq j, k} \lambda_n^2 = B$.

Now consider three coefficients λ_j , λ_k and λ_l such that $\lambda_j \neq \lambda_k$ and $\lambda_j \neq \lambda_l$. From Eq. (33) we have $\lambda_j \lambda_k \lambda_l^2 \prod_{n \neq j, k, l} \lambda_n^2 = B$ and $\lambda_j \lambda_l \lambda_k^2 \prod_{n \neq j, k, l} \lambda_n^2 = B$, so

that the left-hand sides must agree, from which it follows that $\lambda_k = \lambda_l$. Therefore, there can be at most two different values for the λ_n ,

$$\lambda_1 = \dots = \lambda_m = \alpha , \quad \lambda_{m+1} = \dots = \lambda_d = \beta . \quad (34)$$

By inserting this into the conditions (30), (31) one obtains

$$m\alpha + (d-m)\beta = \sqrt{d F_\psi} \quad (35)$$

$$m\alpha^2 + (d-m)\beta^2 = 1 . \quad (36)$$

For given m , those equations are solved by

$$\alpha_{\pm} = \sqrt{\frac{F_\psi}{d}} \pm \sqrt{\frac{d-m}{m}} \sqrt{\frac{1-F_\psi}{d}} \quad (37)$$

$$\beta_{\pm} = \sqrt{\frac{F_\psi}{d}} \mp \sqrt{\frac{m}{d-m}} \sqrt{\frac{1-F_\psi}{d}} . \quad (38)$$

For $m = 0$ and $m = d$ there is no general solution; further, we see that it is sufficient to consider $m < \frac{d}{2}$. We know from the axisymmetric states that $F_\psi \geq \frac{d-1}{d}$; what remains to do is to determine the m and the sign for the best lower bound. To this end, we check the derivatives $\frac{d\alpha}{dm}$, $\frac{d\beta}{dm}$ and find that for the '+' sign both α and β have their minimum for maximum m , i.e., $m = d-1$ (whereas the '-' sign gives $m = 1$). Both solutions can be mapped to one another. By choosing the '-' sign and $m = 1$ we find Eq. (7), as well as the corresponding α and β .

Concavity of the pure-state minimum

In this section we prove the concavity of Eq. (7). We have

$$C_G = d (\alpha \beta^{d-1})^{\frac{2}{d}} \quad (39)$$

$$= d \left(\frac{1}{B} - 1 \right)^{\frac{1}{d}} B (d-1)^{\frac{1}{d}-1} , \quad (40)$$

where

$$\begin{aligned} B &:= (d-1)\beta^2 \\ &= \frac{1}{d} \left(1 + (d-2)F + 2\sqrt{d-1}\sqrt{F(1-F)} \right) . \end{aligned} \quad (41)$$

The last line here is obtained by substituting $\alpha(F)$, $\beta(F)$, see Eq. (7).

The second derivative of C_G with respect to F is (up to constant positive prefactors for fixed dimension $d > 1$)

$$\begin{aligned}
\frac{d^2 C_G}{dF^2} &\propto \frac{d^2 B}{dF^2} \left(\frac{1}{B} - 1 \right)^{\frac{1}{d}-1} \left(\frac{d-1}{d} \frac{1}{B} - 1 \right) + \\
&+ \left(\frac{dB}{dF} \right)^2 \left(\frac{d-1}{d} \frac{1}{B} - 1 \right) \left(\frac{1}{d} - 1 \right) \left(\frac{1}{B} - 1 \right)^{\frac{1}{d}-2} \left(-\frac{1}{B^2} \right) + \\
&+ \left(\frac{dB}{dF} \right)^2 \left(\frac{1}{B} - 1 \right)^{\frac{1}{d}-1} \frac{d-1}{d} \left(-\frac{1}{B^2} \right) \\
&= \frac{1}{d^2 B^3} \left(\frac{1}{B} - 1 \right)^{\frac{1}{d}-2} \left[\frac{d^2 B}{dF^2} dB(1-B)(d-1-dB) - \left(\frac{dB}{dF} \right)^2 (d-1) \right]. \quad (42)
\end{aligned}$$

As the prefactor in the last line is positive we only need the sign of the term in square brackets [...] in order to decide about the sign of the second derivative. Now we use the explicit expression for $B(F)$ in Eq. (41) to calculate the derivatives with respect to F . After some algebra we find for the square bracket in Eq. (42)

$$\begin{aligned}
[\dots] &= \frac{B(1-B)}{F(1-F)} \times \\
&\times \left[-\frac{\sqrt{d-1}}{2\sqrt{F(1-F)}}(d-1-dB) - (d-1) \right]. \quad (43)
\end{aligned}$$

Again, the first factor is positive. Now we substitute the expression for $B(F)$, Eq. (41), in $(d-1-dB)$ which yields

$$[\dots] = \left[-\sqrt{d-1}(d-2)\sqrt{\frac{1-F}{F}} \right] \frac{B(1-B)}{F(1-F)} \leq 0, \quad (44)$$

and thus concludes the proof.

Proof of Eq. (11)

Consider a family of states \mathcal{M} that are invariant under a group G of entanglement-preserving transformations, that is, $\rho_g := g\rho g^{-1} = \rho$ for $g \in G$ and $\rho \in \mathcal{M}$. Let $\mathcal{S}_k^G \subset \mathcal{M}$ be the set of states that are symmetric under G and have Schmidt number k ($1 \leq k < d$), and \mathcal{S}_k the set of all Schmidt number k states. Given an arbitrary state ρ , the minimum distance with respect to the closest Schmidt number k state, σ^* , is

$$\begin{aligned}
\min_{\sigma \in \mathcal{S}_k} \|\rho - \sigma\|_p &= \|\rho - \sigma^*\|_p = \int dg \|\rho_g - \sigma_g^*\|_p \\
&\geq \left\| \int dg (\rho_g - \sigma_g^*) \right\|_p = \|\rho_G - \sigma_G^*\|_p \\
&\geq \min_{\sigma \in \mathcal{S}_k^G} \|\rho_G - \sigma\|_p, \quad (45)
\end{aligned}$$

where we have used the triangle inequality in the second line, $X_G := \int dg g X g^{-1}$, and $\|\cdot\|_p$ is any Schatten p -norm with $1 \leq p \leq \infty$.

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